# Mechanics of Solids: Energy principles

**Work and Energy in linear elasticity**

- Potential energy
- Stability and uniqueness of solution

- Principle of virtual work
- Principle of minimum potential energy
- Examples

From the book: Mechanics of Continuous Media: an Introduction

1. J Botsis and M Deville, PPUR 2018.
2. J Botsis, Appendix A Notes on Energy

# Mechanics of Solids: Equations of Elasticity

**FORMULATION OF THE BOUNDARY VALUE PROBLEM**

We consider a solid, of an isotropic homogeneous linearly elastic material, and subjected to body forces over it and prescribed displacements or tractions on its boundary. The following equations are available:

1. The 3 eqs of equilibrium:

$$\boldsymbol{\sigma}_{ij,j} + f_i = 0 \quad , \quad div\boldsymbol{\sigma} + \boldsymbol{f} = 0$$

($\boldsymbol{f}$ is body force vector)

2. The 6 equations defining the *strain-displacement relation:*

$$\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad , \quad \boldsymbol{\varepsilon} = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$$

3. The 6 equations defining the *isotropic homogeneous stress-strain relation:*

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \quad , \quad \boldsymbol{\sigma} = \lambda tr\boldsymbol{\varepsilon}\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}$$

There are 15 equations with 15 unknowns:

Three displacement components: $u_i$

Six strain components: $\varepsilon_{ij}$

Six stress components: $\sigma_{ij}$

→ The problem is well posed

We know that a linear elastic solid satisfies the second principle of thermodynamics and that there exists a potential function which, has a quadratic form in the strains (or the stresses).

$$W(\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} \quad \text{with} \quad \sigma_{ij} = \frac{\partial W(\boldsymbol{\varepsilon}_{ij})}{\partial\boldsymbol{\varepsilon}_{ij}}$$

# Mechanics of Solids: Navier equations

## Navier's Equations

There are two ways to combine the 15 equations:

The first one is to consider the displacement components $u_i$ as the unknowns.

Introduce $\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ in

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

to obtain:

$$\sigma_{ij} = \lambda u_{k,k}\,\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

Introduce it to the equilibrium equations:

$$\boldsymbol{\sigma}_{ij,j} + f_i = 0$$

to obtain:

$$(\lambda + \mu)u_{k,ki} + \mu u_{i,jj} + f_i = 0$$

These are the three Navier's Equations with the three displacement components $u_i$ as the unknowns.

With the displacements known we go back to:

$$\boldsymbol{\varepsilon}_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

to calculate the strains $\varepsilon_{ij}$

With the strains known we obtain the stresses $\sigma_{ij}$

from $\sigma_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$

Note that there is no need to satisfy the compatibility equations:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0$$

because we calculate $\varepsilon_{ij}$ from $u_i$

# Mechanics of Solids: Beltrami-Michell equations

## BELTRAMI-MICHELL COMPATIBILITY EQUATIONS

There are two ways to combine the 15 equations:

We an consider the stress components $\sigma_{ij}$ as unknowns.

Then we introduce the strain-stress relations:

$$\varepsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}$$

In the compatibility equations

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{jl,ik} - \varepsilon_{ik,jl} = 0$$

to obtain:

$$(1+\nu)\sigma_{ij,kk} - \nu\sigma_{mm,nn}\delta_{ij} + \sigma_{pp,ij}$$
$$-(1+\nu)(\sigma_{iq,qj} + \sigma_{jr,ri}) = 0$$

From the equilibrium equations (take the derivatives):

$$\sigma_{iq,qi} + \sigma_{jr,ri} = -f_{i,j} - f_{j,i}$$

$$(1+\nu)\sigma_{ij,kk} - \nu\sigma_{mm,nn}\delta_{ij} + \sigma_{pp,ij}$$
$$+(1+\nu)(f_{i,j} + f_{j,i}) = 0$$

Taking the trace of the last equation we get:

$$(1-\nu)\sigma_{mm,nn} = -(1+\nu)f_{k,k}$$

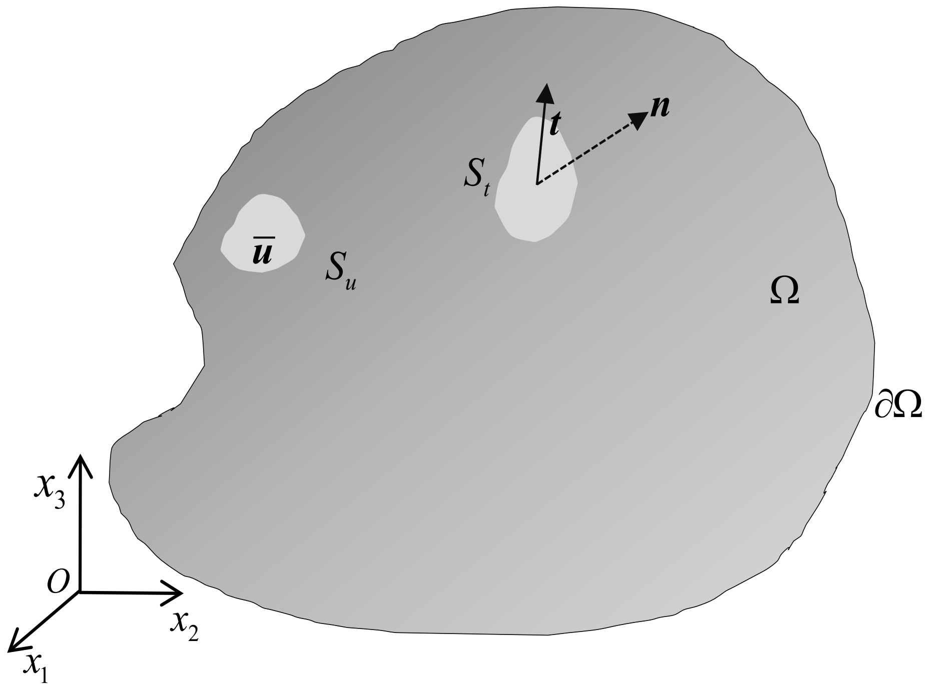
Using it in the last equation we obtain $(\nu \neq 1)$ the Beltrami-Michell compatibility Eqs:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu}f_{n,n}\delta_{ij} = 0$$

In several problems the body forces can be assumed negligible. We have the simplification:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} = 0$$

# Mechanics of Solids: Boundary Value Problems in elasticity




**type I,** or **mixed BVP**: we have to specify tractions and displacements on the corresponding parts of boundaries.

**type II**: we have to specify displacement on the Corresponding boundary conditions.

**type III:** we have to specify tractions on the corresponding part of boundaries:

**BOUNDARY CONDITIONS**

To solve the system of equations we need the appropriate boundary conditions: In general we have three of them.

We consider a body occupying a domain $\Omega$ in $\mathbb{R}^3$ with boundary $\partial\Omega$.

We divide the surface boundary into two parts so that:

$$\partial\Omega = S_u \cup S_t \quad , \quad S_u \cap S_t = \varnothing$$

$S_u$ represents the part where displacements are prescribed:

$$u_i = \overline{u}_i \quad \text{on} \quad S_u$$

$S_t$ represents the part where stress vector is prescribed:

$$t_i = \boldsymbol{\sigma}_{ij} n_j = \bar{t}_i \quad \text{on} \quad S_t$$

**TYPE I** or **mixed BVP**: we have to specify tractions and displacements on the corresponding parts of boundaries:

Navier Equations to solve $(\lambda+\mu)u_{k,ki}+\mu u_{i,jj}+f_i=0$ over $\Omega$

Subjected to Boundary Conditions:

Tractions: $t_i=\sigma_{ij}n_j=\overline{t_i}$ on $S_t$

$\sigma_{ij}=\lambda u_{k,k}\delta_{ij}+\mu(u_{i,j}+u_{j,i})$ ⇒ $\lambda u_{k,k}n_i+\mu(u_{i,j}+u_{j,i})n_j=\overline{t_i}$ on $S_t$

Displacements: $u_i=\overline{u_i}$ on $S_u$

# Mechanics of Solids: Boundary Value Problems in elasticity

**TYPE II**: Displacement Boundary Conditions

we only have displacement boundary conditions $S_u \neq \varnothing \quad , \quad S_t = \varnothing$

Navier Equations to solve $(\lambda + \mu) u_{k,ki} + \mu u_{i,jj} + f_i = 0$ over $\Omega$

Subjected to Boundary Conditions

Displacements: $u_i = \overline{u}_i$ on $S_u$

# Mechanics of Solids: Boundary Value Problems in elasticity

**TYPE III:** we have to specify tractions on the corresponding part of boundaries:

Navier Equations to solve $(\lambda+\mu)u_{k,ki}+\mu u_{i,jj}+f_i=0$ over $\Omega$

Subjected to Boundary Conditions

Tractions: $t_i=\sigma_{ij}n_j=\overline{t_i}$ on $S_t$

$\sigma_{ij}=\lambda u_{k,k}\delta_{ij}+\mu(u_{i,j}+u_{j,i})$ ⟹ $\lambda u_{k,k}n_i+\mu(u_{i,j}+u_{j,i})n_j=\overline{t_i}$ on $S_t$

# Mechanics of Solids: Boundary Value Problems in elasticity

**The traction BVP in terms of stress components :** $S_u = \varnothing \quad , \quad S_t \neq \varnothing$

Here the following equations constitute the problem:

1. Equations equilibrium

$$\sigma_{ij,j} + f_i = 0 \quad \text{over} \quad \Omega$$

2. Stress compatibility equations:

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{mm,ij} + f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu} f_{n,n}\delta_{ij} = 0 \quad \text{over} \quad \Omega$$

3. Prescribed tractions on the surface:

$$t_i = \sigma_{ij} n_j = \overline{t_i} \quad \text{on} \quad S_t$$

**Theorem of Work and Energy**:

Multiply the equations of Equilibrium :

$$\sigma_{ij,j} + f_i = 0$$

by the displacement $u_i$ and integrate over volume $\Omega$

$$\int_\Omega \sigma_{ij,j} u_i dv + \int_\Omega f_i u_i dv = 0$$

<u>To proceed we will use the following in the last relation</u> :

1. The symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$
2. The Cauchy formula $t_i = \sigma_{ij} n_j = \overline{t_i}$
3. The relation $(\sigma_{ij} u_i),_j = \sigma_{ij,j} u_i + \sigma_{ij} u_{i,j} \quad \Rightarrow \quad \sigma_{ij,j} u_i = (\sigma_{ij} u_i),_j - \sigma_{ij} u_{i,j} = \sigma_{ij,j} u_i$

**Theorem of Work and Energy:**

$$\int_\Omega \sigma_{ij,j} u_i dv = \int_\Omega \left( (\sigma_{ij} u_i)_{,j} - \sigma_{ij} u_{i,j} \right) dv = \int_\Omega \left( (\sigma_{ij} u_i)_{,j} \right) dv - \int_\Omega \left( \sigma_{ij} u_{i,j} \right) dv$$

$$= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left( \frac{1}{2} (\sigma_{ij} u_{i,j} + \sigma_{ij} u_{i,j}) \right) dv$$

$$= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left( \frac{1}{2} (\sigma_{ij} u_{i,j} + \sigma_{ji} u_{j,i}) \right) dv$$

$$= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left( \frac{1}{2} (\sigma_{ij} u_{i,j} + \sigma_{ij} u_{j,i}) \right) dv$$

$$= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \left( \sigma_{ij} \frac{1}{2} (u_{i,j} + u_{j,i}) \right) dv$$

$$= \int_{\partial\Omega} \sigma_{ij} u_i n_j ds - \int_\Omega \sigma_{ij} \varepsilon_{ij} dv$$

$$= \int_{\partial\Omega} t_i u_i ds - \int_\Omega \sigma_{ij} \varepsilon_{ij} dv.$$

$$\int_\Omega \sigma_{ij,j} u_i dv + \int_\Omega f_i u_i dv = 0$$

$$\int_\Omega \sigma_{ij} \varepsilon_{ij} dv = \int_{\partial\Omega} t_i u_i ds + \int_\Omega f_i u_i dv$$

**Theorem of Work and Energy**:

$$\int_{\Omega} \sigma_{ij}\varepsilon_{ij} dv = \int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv$$

Use $\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

and recall $W(\varepsilon_{ij}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij}$

in the integral on the left hand side

$$\int_{\Omega} \sigma_{ij}\varepsilon_{ij} dv = \int_{\Omega} \left(\lambda\varepsilon_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ij}\varepsilon_{ij}\right) dv = 2\int_{\Omega} W(\varepsilon_{ij}) dv$$

$$\frac{1}{2}\left(\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv\right) = \int_{\Omega} W(\varepsilon_{ij}) dv$$

the *theorem of work and energy,*
or principle of conservation of
mechanical energy,
for an isotropic linearly elastic solid.

# Mechanics of Solids: Energy Principles

**Potential Energy:** $\Pi$

It is defined as the difference of

Strain Energy: $U$

Work of the applied forces: $\mathbb{W}$

$$\Pi = U - \mathbb{W}$$

$$\frac{1}{2}\left(\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv\right) = \int_{\Omega} W(\varepsilon_{ij}) dv$$

$$\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv = 2\int_{\Omega} W(\varepsilon_{ij}) dv$$

$$U = \int_{\Omega} \frac{1}{2}\sigma_{ij}\varepsilon_{ij} dv = \int_{\Omega} W(\varepsilon_{ij}) dv$$

$$\mathbb{W} = \int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv = 2U$$

$$\Pi = U - \mathbb{W} = \int_{\Omega} W(\varepsilon_{ij}) dv - \left(\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv\right)$$

$$\Pi = U - \mathbb{W} = U - 2U = -U = -\frac{1}{2}\mathbb{W}$$

It is an important energy function in predicting equilibrium and modeling phenomena such as fracture.

$\int_\Omega (9K\varepsilon_{ii}\varepsilon_{kk} + 2\mu\varepsilon_{ij}^d\varepsilon_{ij}^d)dv = 0$

# Mechanics of Solids: Energy Principles

**Strain Energy 'partition'**:

Use the following deviatoric components

$$\sigma_{ij} = \sigma_{ij}^d + \sigma_0\delta_{ij} \quad \sigma_0 = \frac{1}{3}\sigma_{kk} \quad \sigma_{ij}^d = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^d + \varepsilon_0\delta_{ij} \quad \varepsilon_0 = \frac{1}{3}\varepsilon_{kk} \quad \varepsilon_{ij}^d = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \,.$$

The strain energy density is expressed as:

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\lambda(\varepsilon_{kk})^2 + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{9}{2}K(\varepsilon_0)^2 + \mu\varepsilon_{ij}^d\varepsilon_{ij}^d$$

Energy for volume changes

Energy for shape changes

Stability condition by definition

$$W(\boldsymbol{\varepsilon}) > 0 \ \forall \boldsymbol{\varepsilon} \neq \mathbf{0}$$

$$K > 0 \text{ and } \mu > 0$$

**Uniqueness of solution**:

Consider two solutions

$\left(\boldsymbol{u}^{(1)}, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\sigma}^{(1)}\right) \quad \left(\boldsymbol{u}^{(2)}, \boldsymbol{\varepsilon}^{(2)}, \boldsymbol{\sigma}^{(2)}\right)$

satisfying the filed equations

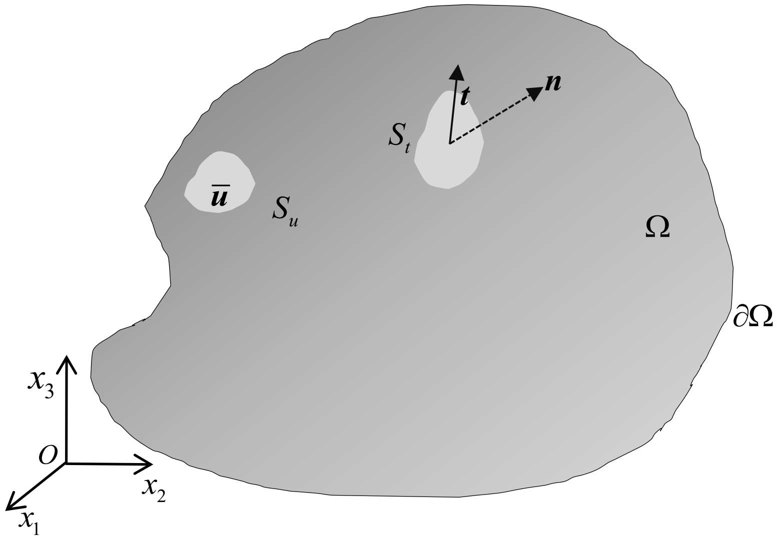
$$\sigma_{ij,j} + f_i = 0; \ \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right);$$

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

and the boundary conditions

$$u_i = \overline{u}_i \quad on \quad S_u$$

$$t_i = \sigma_{ij} n_j = \overline{t}_i \quad on \quad S_t$$




$$if \quad S_u \neq \varnothing \quad \Rightarrow \boldsymbol{u}^{(2)} = \boldsymbol{u}^{(1)}, \quad \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(1)}, \quad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)}$$

$$if \quad S_u = \varnothing \quad \Rightarrow \boldsymbol{u}^{(2)} = \boldsymbol{u}^{(1)} + \boldsymbol{w}, \quad \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{(1)}, \quad \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)}$$

( $\boldsymbol{w}$ is an infinitesimal rigid body rotation).

## Uniqueness of solution

Define the differences:

$$\boldsymbol{u} = \boldsymbol{u}^{(2)} - \boldsymbol{u}^{(1)}, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(2)} - \boldsymbol{\varepsilon}^{(1)}, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{(2)} - \boldsymbol{\sigma}^{(1)}$$

$\Rightarrow \ \boldsymbol{u}, \ \boldsymbol{\varepsilon}, \ \boldsymbol{\sigma}$ satisfy the equations with $f_i = 0$;

$$\sigma_{ij,j} + f_i = 0; \ \sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right);$$

and the boundary conditions

$$u_i = 0 \ \ on \ \ S_u \quad , \quad t_i = \sigma_{ij} n_j = 0 \ \ on \ \ S_t$$

From energy we have

$$\int_{\partial\Omega} t_i u_i ds + \int_{\Omega} f_i u_i dv = \int_{\Omega} \left( 9K \left( \varepsilon_0 \right)^2 + 2\mu \varepsilon_{ij}^d \varepsilon_{ij}^d \right) dv$$

$$u_i = 0 \ ; \ t_i = \sigma_{ij} n_j = 0$$

$$\int_{\Omega} \left( K \varepsilon_{ii} \varepsilon_{kk} + 2\mu \varepsilon_{ij}^d \varepsilon_{ij}^d \right) dv = 0$$

Because $K > 0$ and $\mu > 0$

$$\varepsilon_{ij} = \varepsilon_{ij}^d = 0$$

$$\Rightarrow \sigma_{ij} = 0$$

# Mechanics of Solids: Energy Principles

**Theorem of Virtual Work**

*Virtual displacement:*

- *It is an arbitrary displacement which does not affect the force system acting on the body during its application.*
- *Its components* $\delta \boldsymbol{u}\ (\delta u_1, \delta u_2, \delta u_3)$ *are small, continuous and single valued.*
- *All forces remain constant in magnitude and direction during application of virtual displacement.*

The virtual displacement satisfies:

$$\delta u_i = 0 \quad on \quad S_u$$

$$\delta \varepsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})$$

$$\delta \varepsilon_{ij} = \frac{1}{2}\delta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(\delta u_i) + \frac{\partial}{\partial x_i}(\delta u_j)\right)$$

For a body in equilibrium

$$\sigma_{ij,j} + f_i = 0 \quad \text{over} \quad \Omega$$

$$t_i = \sigma_{ij} n_j = \overline{t}_i \quad \text{on} \quad S_t$$

$$u_i = \overline{u}_i \quad \text{on} \quad S_u$$

*Multiply the equilibrium Eqs by* $\delta u_i$ *and follow the same steps as in the case of the theorem of work and energy.*

$$\int_\Omega \sigma_{ij} \delta \varepsilon_{ij} dv = \int_{S_t} \overline{t}_i \delta u_i ds + \int_\Omega f_i \delta u_i dv$$

Virtual strain energy

Virtual work by applied forces

# Mechanics of Solids: Energy Principles

**Principle of minimum potential energy**

*of all displacement fields satisfying the continuity and boundary conditions, of the solid in equilibrium, the actual displacement field makes the potential energy a stationary value*.

From the principle of virtual work we have:

$$\delta U = \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}dv = \int_{S_t} \bar{t}_i \delta u_i ds + \int_\Omega f_i \delta u_i dv$$

$$\delta \mathbb{W} = \int_{\partial\Omega} \sigma_{ij} n_j \delta u_i ds + \int_\Omega f_i \delta u_i dv$$

With $\Pi = U - \mathbb{W}$

$$\delta\Pi = \delta\left(U - \mathbb{W}\right) =$$

$$= \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}dv - \left(\int_{\partial\Omega} \sigma_{ij} n_j \delta u_i ds + \int_\Omega f_i \delta u_i dv\right) = 0$$

# Mechanics of Solids: Energy Principles

For a body in equilibrium

$$\sigma_{ij,j} + f_i = 0 \quad \text{over} \quad \Omega$$

$$t_i = \sigma_{ij} n_j = \bar{t}_i \quad \text{on} \quad S_t$$

$$u_i = \bar{u}_i \quad \text{on} \quad S_u$$

A displacement field $\tilde{\boldsymbol{u}}$ is *kinematically admissible* if it respects the assigned displacements on the boundary:

$$\tilde{u}_i = \bar{u}_i \quad \text{on} \quad S_u$$

**Theorem of minimum potential energy**

*For a body in equilibrium and having a solution* $u_i$

*For any kinematically admissible displacement* $\tilde{u}_i$ *the potential energies satisfy the condition*

$$\Pi(u) \leq \Pi(\tilde{u}_i)$$

*Define* $\delta u_i = \tilde{u}_i - u_i$

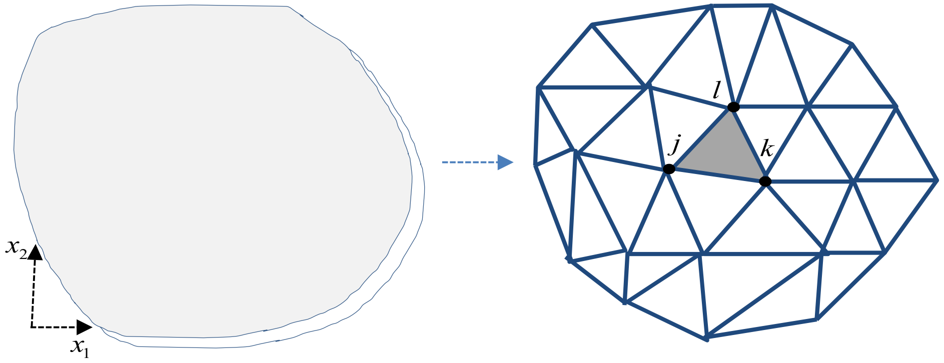
*Use the principle of virtual work to obtain:*

$$\Pi(u_i + \delta u_i) - \Pi(u_i) = \int_\Omega \frac{1}{2} \frac{\partial^2 W(\varepsilon_{ij})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \delta\varepsilon_{ij} \delta\varepsilon_{kl} dv \geq 0$$

# Mechanics of Solids: Energy Principles

## **Application to Finite Elements**:

A body is discretized with *m* triangular elements




**For each element define**

1. Nodal displacements:

$$\left(\tilde{\delta}\right)_e = \left(u_1^j, u_2^j, u_1^k, u_2^k, u_1^l, u_2^l\right)$$

2. Displacement functions:

$$\left(\tilde{f}\right)_e = \left(u_i\left(x_1, x_2\right), u_2\left(x_1, x_2\right)\right)$$

3. Strain displacement relations:

$$\left(\varepsilon\right)_e = \left[B\right]\left(\tilde{\delta}\right)_e$$

4. Constitutive relations for each element:

$$\left(\sigma\right)_e = \left[C\right]\left(\varepsilon\right)_e$$

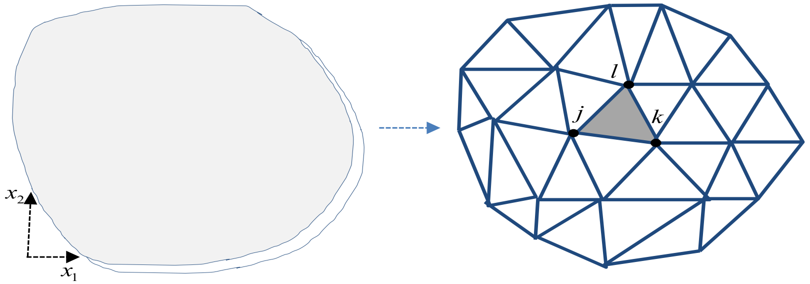
Principle of minimum potential energy for the entire body:

$$\sum_1^m \int_{\Omega_e} \sigma_{ij}\delta\varepsilon_{ij} dv - \sum_1^m \int_{\partial\Omega_e} \overline{t}_i \delta u_i ds - \sum_1^m \int_{\Omega_e} f_i \delta u_i dv = 0$$

# Mechanics of Solids: Energy Principles

## **Application to Finite Elements**:

A body is discretized with *m* triangular elements




Principle of minimum potential energy for the Entire body:

$$\sum_1^m \int_{\Omega_e} \sigma_{ij} \delta\varepsilon_{ij} dv - \sum_1^m \int_{\partial\Omega_e} \overline{t}_i \delta u_i ds - \sum_1^m \int_{\Omega_e} f_i \delta u_i dv = 0$$

We introduce the displacements, strain-displacements and constitutive equations to obtain

$$\sum_1^m \left(\delta\tilde{\delta}\right)_e^T \left[ [k]_e \left(\tilde{\delta}\right)_e - (Q)_e \right] = 0$$

For each element we have

$$[k]_e \left(\tilde{\delta}\right)_e = (Q)_e$$

For the entire body we have

$$\left(\delta\tilde{\delta}\right)^T \left[ [K]\left(\tilde{\delta}\right) - (Q) \right] = 0$$

$$\Rightarrow [K]\left(\tilde{\delta}\right) = (Q)$$

with

$$[K] = \sum_1^m [k]_e \quad (Q) = \sum_1^m (Q)_e$$